

# A six dimensional analysis of Maxwell's Field Equations

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A framework based on an extension of Kaluza's original idea of using a five dimensional space to unify gravity with electromagnetism is used to analyze Maxwell's field equations. The extension consists in the use of a six dimensional space in which all equations of electromagnetism may be obtained using only Einstein's field equation. Two major advantages of this approach to electromagnetism are discussed, a full symmetric derivation for the wave equations for the potentials and a natural inclusion of magnetic monopoles without using any argument based on singularities.

## I. INTRODUCTION

In a recent paper [1], we have discussed the method proposed by Kaluza in 1921 [2] to derive Maxwell's equations for an electromagnetic field in vacuum by treating electric charges in space as sources of curvature as mass does in Einstein's theory of general relativity. In Kaluza's original paper, whereas the equations with sources follow directly from the field equations, the homogeneous ones are derived from a geometrical identity which happens to yield the correct results only with a cartesian metric. For generalized coordinates, this last result ceases to hold true. This was specifically illustrated using a Minkowsky space with spherical symmetry. Moreover, we showed further how to overcome the difficulty by introducing a 6 X 6 metric in which the sixth row and columns contain additional potentials ( $Z_i, i = 1, 2, 3; \frac{n}{c}$ ).  $g_{66}$  was taken equal to one and the elements  $g_{56}$  and  $g_{66}$  later proved to be related with two vectors  $M^\mu$  and  $Q^\mu$ , both irrotational, and the full symmetric expressions for Maxwell's equations in terms of magnetic point charges were recovered. The existence of these latter ones, although still controversial, arise in a complete natural way in the formalism.

It was also mentioned in paper [1] that with the results obtained therein, one can also arrive, correspondingly, to a set of completely symmetric wave equations for the field potentials, thus removing the old difficulty of being able to express the magnetic field as the curl of a vector potential.

Indeed, if magnetic monopoles are accounted for, the divergence of the magnetic field cannot vanish. This equation has led to different approaches starting from the pioneering work of Dirac [3] who proposed the idea of considering a singularity along an axis with the monopole at the origin, so the magnetic field can still be taken as a curl in the vector potential in every region of space not containing such line. This idea, as well as other approaches have been discussed by many other authors [4], all of them foreign to Kaluza's ideas.

What we explicitly want to show in this paper is that in the 6D formalism presented in Ref. [1], the two additional potentials lead to two equivalent formulations for the electromagnetic field, and therefore, such potentials can be treated in a completely symmetrical way. It is worth noticing that a similar four potential has been considered before [14], but the expressions for the electromagnetic fields differ from those obtained within Kaluza's framework.

We also want to stress that the inclusion of magnetic monopoles within Kaluza's scheme is not new. It has been done by several authors in different contexts using five dimensions and associating singularities with monopoles [12]. What is entirely new in this treatment is the use of a 6D metric which allows the introduction of monopoles keeping all symmetries between the fields and the potentials.

In section II we briefly recapitulate the results of Ref. [1], in section III we discuss the wave equations for the additional potentials and derive the continuity equations for electric and magnetic charges. We finally leave section IV for some pertinent concluding remarks.

## II. GENERAL BACKGROUND

As it has been discussed at length in the literature, physicists have long searched for magnetic monopoles. The main reason is very simple: *symmetry*. Electromagnetic theory would become much more symmetric if terms proportional to magnetic charges and current densities are added to the Maxwell equations. This has been very clearly pointed out in the literature, such equations adopt a very elegant and symmetric structure. But some inconsistencies appear if one turns to work with the electromagnetic potentials. The magnetic field cannot be represented as a curl anymore, since its divergence is now supposed to be proportional to a magnetic charge density. In Jackson's textbook [9], a solution *for a particular problem* is mathematically stated without major obstacles. Some other authors, in the spirit of Jackson, have proposed to represent a magnetic monopole as a singularity in a four dimensional space-time in which the vector potential is discontinuous. Since in this case the potential is not differentiable, the definition of the field tensor, in terms of potentials, ceases to hold. Then, the magnetic field would have a vanishing divergence in all the four dimensional space, except at the monopole's position. This argument solves the problem but introduces an undesirable asymmetry in the definitions of magnetic and electric potentials. An alternative solution to this problem can be found following the line of Kaluza's theory in a 6D space time. In order to include a magnetic charge and preserve full symmetry in electromagnetic theory, five dimensions are not enough. The fifth dimension introduced by Kaluza was characterized by the condition  $\frac{dx^5}{dt} = \frac{q}{m}$ ,  $q$  being the charge,  $m$  its mass and the cylindrical condition  $\frac{\partial}{\partial x^5} = 0$ . Analogously, a sixth dimension is now introduced according to the conditions

$$\frac{dx^6}{dt} = \frac{g}{m} \quad (1)$$

$$\frac{\partial}{\partial x^6} = 0 \quad (2)$$

where in Eq. (1),  $g$  is the magnetic charge and  $m$  is the rest mass of the particle.

For the sake of simplicity, the metric tensor to work with will consist in a 6D generalization of Minkowsky's space time with the new entries:

$$g_{n6} = g_{6n} = Z_n \quad n = 1, 2, 3 \quad (3)$$

and

$$g_{46} = g_{64} = \frac{\eta}{c} \quad (4)$$

Here, the quantities  $Z_n$  and  $\eta$  will be interpreted later. At present we shall regard them as two potentials, one of scalar type  $\eta$  and one of a vector type  $Z_n$ . This is done in order to preserve symmetry with the conventional theory. For the time being, the coefficients  $g_{56} = g_{65}$  are left undetermined, but will be accounted in a rigorous way later. With the proposed metric, the Christoffel symbols become,

$$\Gamma_{\beta 5}^\alpha = A_{,\beta}^\alpha - A_{,\alpha}^\beta + Z^\alpha g_{56,\beta}, \quad (5)$$

and

$$\Gamma_{\beta 6}^\alpha = Z_{,\beta}^\alpha - Z_{,\alpha}^\beta + A^\alpha g_{56,\beta}. \quad (6)$$

where  $A^n$  is the ordinary vector potential. The Christoffel symbols may also be obtained by requiring that a magnetic field and an electrically charged particle, as well, move along geodesics in this space-time, and comparing its equation of motion with the Lorentz's force, which now includes a symmetric term depending on  $g$ . An example of this procedure was worked out by Stephani [18] in a pure gravitational context. Thus, the proposed Lorentz force reads:

$$\frac{d^2 x^\alpha}{dt^2} = \frac{q}{m} \left[ \varepsilon_{\beta\gamma}^\alpha \frac{\partial x^\beta}{\partial t} B^\gamma + E^\alpha \right] + \frac{g}{m} \left[ \varepsilon_{\beta\gamma}^\alpha \frac{\partial x^\beta}{\partial t} E^\gamma - B^\alpha \right], \quad (7)$$

where  $\varepsilon_{\beta\gamma}^\alpha$  is Levi Civita's tensor. The Christoffel symbols become those of the five dimensional theory, plus the new ones corresponding to the sixth dimension namely,

$$\Gamma_{6n}^m = \varepsilon_{sn}^m E^s,$$

$$\Gamma_{64}^n = -B^n.$$

Comparing both sets of values for the Christoffel symbols (see Appendix A), the definitions of the electric and magnetic fields are obtained. Because there are two sets of Christoffel symbols, we obtain two equivalent forms of expressing the electromagnetic field namely,

$$E^k = -\varphi^{,k} - A_{,4}^k, \quad (8)$$

$$B^k = \eta^{,k} - Z_{,4}^k, \quad (9)$$

$$E^k = \varepsilon_{lm}^k Z^{l,m} + M^k, \quad (10)$$

$$B^k = \varepsilon_{lm}^k A^k + Q^k. \quad (11)$$

If the vectors  $M^k, Q^k$  are defined as:

$$M^k = \begin{bmatrix} Z^2 g_{65,3} \\ Z^3 g_{65,1} \\ Z^1 g_{65,2} \end{bmatrix} \quad Q^k = \begin{bmatrix} A^2 g_{56,3} \\ A^3 g_{56,1} \\ A^1 g_{56,2} \end{bmatrix}, \quad (12)$$

both fields can be represented in two different ways, and become completely symmetric. Here  $Z^n$  and  $A^n$  with  $n=1,2,3$  are the electric and magnetic vector potentials respectively. The new set of definitions account for a duality in the components of the electromagnetic field. With this dual conception of the field tensor, all the theory can be reformulated including magnetic monopoles and wave equations for the potentials can be obtained. This was shown in Ref. [1].

Eqs. (10) and (11) show the importance of the quantities  $g_{56}$  and  $g_{65}$  which are postulated to be independent of time and equal to each other (the metric tensor is symmetric). Then, introducing expressions (10) and (11) in Maxwell's equations for the divergence of the fields (Eqs. (22) and (23)), the relation between the new vectors  $M^k$  and  $Q^k$  is given by:

$$\nabla \cdot \mathbf{Q} = \frac{\rho}{\varepsilon_0} \frac{q}{m}, \quad (13)$$

$$\nabla \cdot \mathbf{M} = \frac{\rho}{\varepsilon_0} \frac{g}{m}, \quad (14)$$

The vectors  $\mathbf{M}$  and  $\mathbf{Q}$  in this equations contain space derivatives of  $g_{56} = g_{65}$ . If this quantity is proposed as a constant value, then *both* electric and magnetic fields would have zero divergence. In such a case, neither magnetic *nor* electric charges would fit in the theory. Note that these vectors are irrotational, a fact that is sustained by a generalized Helmholtz theorem [19].

The role of these vectors becomes transparent when the new definitions for the fields are replaced in the electromagnetic vector equations. The full implications of this step will be discussed in the following section.

As it was mentioned in the introduction, there is an inherent asymmetry in the way used by Kaluza to derive Maxwell's equations. The inhomogeneous equations are derived from the field equation. The other two equations are a consequence of making the cylindrical condition act over an expression that is identically zero by itself *so no additional information can be extracted from it*. Instead, when the theory is formulated in 6 dimensions, *the field equation can be used twice*, once for each extra dimension and therefore both pairs of equations can be obtained. In order to analyze this last statement, we start from a stress tensor, which reads:

$$T^{5\nu} = \rho v^5 v^\nu, \quad (15)$$

$$T^{6\nu} = \rho v^6 v^\nu. \quad (16)$$

Here  $\rho$  is the rest mass density and  $v^\nu = \frac{dx^\nu}{ds}$  with  $\mu, \nu$  running from one to six. Ricci's tensor is calculated by contracting the Riemann-Christoffel curvature tensor and, considering the curvature scalar equal to zero, the field equation becomes

$$G_{\mu\nu} = R_{\mu\nu} = \kappa T_{\mu\nu} \quad (17)$$

The stress tensor includes now several components porportional to the electric and magnetic current densities. Using  $v^5 = \frac{q}{m}$  and  $v^6 = \frac{g}{m}$  we have that,

$$R_{\mu 5} = \kappa T_\mu = \alpha J_\mu \quad (18)$$

$$R_{\mu 6} = \kappa T_{\mu 6} = \alpha K_\mu \quad (19)$$

where  $J^\mu = \rho \frac{q}{m} v^\mu$  and  $K^\mu = \rho \frac{g}{m} v^\mu$  with  $\mu = 1, 2, 3, 4$ .

Making  $\mu = 1, 2, 3$ , expressions (18) and (19) become, in vector form,

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} - \mathbf{K} \quad (20)$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \quad (21)$$

and if we take  $\mu = 4$ , in those same equations, the following well-known expressions are obtained:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \frac{q}{m} \quad (22)$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho \frac{g}{m} \quad (23)$$

In the absence of magnetic charge ( $g = 0$ ), the usual electromagnetic equations are recovered. In the presence of magnetic charges, the stress tensor has new components which, by means of the field equation, lead to a full symmetric set of electromagnetic equations.

### III. WAVE EQUATIONS FOR ELECTROMAGNETIC POTENTIALS

After the lengthy but necessary account of the main results required to tackle the crucial question in this paper, we will now assess the usefulness of the 6D space in deriving the wave equation for the new potentials  $\eta$  and  $Z^n$  introduced in Eqs. (3) and (4). without resorting to arguments concerning singularities. In fact, if we now substitute the definitions of the vectors  $E^k$ ,  $B^k$  given in Eqs. (8-9) into Eqs. (22-23), one finds that

$$\nabla^2 \eta + \frac{\partial}{\partial t} \nabla \cdot \mathbf{Z} = \mu_0 \rho \frac{g}{m}. \quad (24)$$

It is important to notice that in order to obtain the wave equations, it becomes necessary to introduce one additional fact, namely, the condition that the Lorentz's gauge  $A^\mu_{,\mu} = 0$  has to be also applied to the new vector potential  $Z^n$ , so that  $Z^\mu_{,\mu} = 0$ .

This condition implies

$$\nabla \cdot \mathbf{Z} + \frac{1}{c^2} \frac{\partial \eta}{\partial t} = 0, \quad (25)$$

which, when replaced in eq. (24), gives

$$\nabla^2 \eta - \frac{1}{c^2} \frac{\partial^2 \eta}{\partial t^2} = \mu_0 \rho \frac{g}{m}. \quad (26)$$

On the other hand, the alternative definition for  $\mathbf{B}$  given in eq. (11) and the usual one for  $\mathbf{E}$  (8) can be replaced in equation (21), so that

$$\nabla \times (\nabla \times \mathbf{A} + \mathbf{Q}) = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (-\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t}), \quad (27)$$

which is easily reduced to the result that

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \nabla \times \mathbf{Q} = \mu_0 \mathbf{J} - \frac{1}{c^2} \nabla \frac{\partial \varphi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}. \quad (28)$$

Recalling that  $\mathbf{Q}$  is irrotational (see section 4) and using the gauge condition  $A_{,\mu}^\mu = 0$  we can rewrite Eq. (28) as:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{J}. \quad (29)$$

Following an analogous procedure, the equation for  $\mathbf{Z}$  can also be obtained. The result reads:

$$\nabla^2 \mathbf{Z} - \frac{1}{c^2} \frac{\partial^2 \mathbf{Z}}{\partial t^2} = \frac{1}{\varepsilon_0} \mathbf{K}. \quad (30)$$

It is now important to remark that the above procedure could be questioned by inquiring what would happen if the complementary definitions for  $\mathbf{E}$  and  $\mathbf{B}$  are introduced in the same field equation (20) or (21). To examine this puzzle let us now take  $\mathbf{E}$  defined by Eq. (8) and  $\mathbf{B}$  by Eq. (11) and substitute them into Eq. (20). This will give rise to the following result,

$$\nabla \times (-\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A} + \mathbf{Q}) - \frac{1}{\varepsilon_0} \mathbf{K}, \quad (31)$$

and therefore

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{1}{\varepsilon_0} \mathbf{K} = 0. \quad (32)$$

Moreover, introducing the definition given by eq. (11) for  $\mathbf{B}$  in equation (23) and taking divergence on both sides of it, the following expressions are obtained:

$$\nabla \cdot (\nabla \times \mathbf{A} + \mathbf{Q}) = \mu_0 \rho \frac{g}{m}, \quad (33)$$

or,

$$\nabla \cdot \mathbf{Q} = \mu_0 \rho \frac{g}{m}. \quad (34)$$

Taking the time derivative of (34), the divergence in (32) and combining the resulting equations, a continuity equation for the magnetic charge is obtained, namely:

$$\nabla \cdot \frac{\partial \mathbf{Q}}{\partial t} = \mu_0 \frac{\partial(\rho \frac{g}{m})}{\partial t}, \quad (35)$$

or, in a more familiar notation,

$$\frac{1}{c^2} \frac{\partial(\rho \frac{g}{m})}{\partial t} + \nabla \cdot \mathbf{K} = 0. \quad (36)$$

Therefore, such a procedure generates the continuity equations, both for magnetic as well as electric charges in terms of their respective fluxes,  $\mathbf{K}$  and  $\mathbf{J}$ . Thus the symmetry sought for the electromagnetic field equations is complete.

Summarizing, in this section, wave equations for potentials have been obtained. These equations are completely symmetric. When the new definitions are introduced into Maxwell's equations, wave equations for potentials or continuity equations for the charges are obtained.

#### IV. CONCLUDING REMARKS

Under the assumption that assures the existence of magnetic monopoles, working in a six-dimensional space-time gives useful information about the nature of electromagnetic fields and potentials. In the traditional theory, the expressions for the electric and magnetic field in terms of potentials are different. This difference is caused by the absence of a magnetic charge in the formalism and leads to an asymmetric set of Maxwell's equations. By means of an

extension of Minkowski's metric to six dimensions, symmetric Maxwell's equations result from Einstein's field equation. As part of the geometrical description inherent to this six dimensional space-time, an electric vector potential and a magnetic scalar potential are required. With this set of functions, *dual definitions for the fields can be formulated*, and, with them, four wave equations, one for each potential,  $\eta$ ,  $\varphi$ ,  $\mathbf{A}$  and  $\mathbf{Z}$  are obtained.

Magnetic monopoles are theoretically possible but none has been detected so far. Living in a big universe induces people to think that the possibility of its existence is not remote. Research is being done in this line. Therefore, since no magnetic charges are found yet, it is not possible to estimate the order of the charge-mass ratio of a magnetic monopole, without using quantum theory. This ratio shall characterize the sixth dimension by being equal to it's time variation, as is expressed by Eq.(1). Meanwhile, the assumption associating a small value for  $g/m$  seems to be correct, but this hypothesis certainly needs a stronger support.

This paper establishes something more than a different formulation of results already obtained in magnetic monopole physics. The theory of a six dimensional space-time gives additional information on the subject. The duality of the field tensor has been mentioned before [19] but in this work a natural origin of this property is exhibited. Six dimensions implies two four-potentials, and so, two different symmetric definitions for each field. Therefore, an electromagnetic theory including magnetic monopoles, formulated in a six-dimensional space-time, gets closer to the challenge of a universal description of all phenomena in nature based in the geometry of space-time.

## APPENDIX A: CHRISTOFFEL SYMBOLS AND ITS RELATION WITH THE ELECTROMAGNETIC FIELD

There are two different mechanisms by which Christoffel symbols can be obtained. The first one consists in comparing the equation for a geodesic in a six dimensional space with a corresponding equation of motion for a charged particle under the influence of a generalized Lorentz's force. This equation of motion reads

$$\frac{d^2 x^\alpha}{dt^2} = \frac{q}{m} \left[ \varepsilon_{\beta\gamma}^\alpha \frac{\partial x^\beta}{\partial t} B^\gamma + E^\alpha \right] + \frac{g}{m} \left[ \varepsilon_{\beta\gamma}^\alpha \frac{\partial x^\beta}{\partial t} E^\gamma - B^\alpha \right]. \quad (\text{A1})$$

For example, let's take the first component of equation (A1):

$$\frac{d^2 x^1}{dt^2} = \frac{q}{m} \frac{\partial x^2}{\partial t} B_3 - \frac{q}{m} \frac{\partial x^3}{\partial t} B_2 + \frac{q}{m} E_1 + \frac{g}{m} \frac{\partial x^2}{\partial t} E_3 - \frac{g}{m} \frac{\partial x^3}{\partial t} E_2 - \frac{g}{m} B_1, \quad (\text{A2})$$

which shall be compared, as mentioned before, with the first component of a geodesic, namely,

$$\frac{d^2 x^1}{dt^2} + \Gamma_{5\nu}^1 \frac{q}{m} \frac{dx^\nu}{ds} + \Gamma_{6\nu}^1 \frac{g}{m} \frac{dx^\nu}{ds} = 0. \quad (\text{A3})$$

Equation (A3), after running the indices from 1 to 6, turns out as follows

$$\begin{aligned} \frac{d^2 x^1}{dt^2} + \Gamma_{51}^1 \frac{q}{m} \frac{dx^1}{ds} + \Gamma_{52}^1 \frac{q}{m} \frac{dx^2}{ds} + \Gamma_{53}^1 \frac{q}{m} \frac{dx^3}{ds} + \Gamma_{54}^1 \frac{q}{m} \frac{dx^4}{ds} + \\ \Gamma_{61}^1 \frac{g}{m} \frac{dx^1}{ds} + \Gamma_{62}^1 \frac{g}{m} \frac{dx^2}{ds} + \Gamma_{63}^1 \frac{g}{m} \frac{dx^3}{ds} + \Gamma_{64}^1 \frac{g}{m} \frac{dx^4}{ds} = 0. \end{aligned} \quad (\text{A4})$$

Now, comparing equations (A2) and (A4) one obtains

$$\Gamma_{51}^1 = 0, \quad (\text{A5})$$

$$\Gamma_{52}^1 = B_3, \quad (\text{A6})$$

$$\Gamma_{53}^1 = -B_2, \quad (\text{A7})$$

$$\Gamma_{54}^1 = \frac{1}{c} E_1; \quad (\text{A8})$$

and

$$\Gamma_{61}^1 = 0, \quad (\text{A9})$$

$$\Gamma_{62}^1 = E_3, \quad (\text{A10})$$

$$\Gamma_{63}^1 = -E_2, \quad (\text{A11})$$

$$\Gamma_{64}^1 = -\frac{1}{c}B_1. \quad (\text{A12})$$

The second mechanism to obtain Christoffel symbols consists in introducing the metric elements in their definition, see ref. [1]. The expressions for symbols in equations (A5) to (A12) are given below

$$\Gamma_{51}^1 = \frac{1}{2}g^{11} \left( \frac{\partial g_{51}}{\partial x^1} - \frac{\partial g_{51}}{\partial x^1} \right) + \frac{1}{2}g^{15} \frac{\partial g_{55}}{\partial x^1} = 0, \quad (\text{A13})$$

$$\begin{aligned} \Gamma_{52}^1 &= \frac{1}{2}g^{11} \left( \frac{\partial g_{51}}{\partial x^2} - \frac{\partial g_{52}}{\partial x^1} \right) + \frac{1}{2}g^{16} \frac{\partial g_{56}}{\partial x^2} \\ &= \frac{1}{2} \left( \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} \right) + \frac{1}{2}Z_1 \frac{\partial g_{56}}{\partial x^2}, \end{aligned} \quad (\text{A14})$$

$$\begin{aligned} \Gamma_{53}^1 &= \frac{1}{2}g^{11} \left( \frac{\partial g_{51}}{\partial x^3} - \frac{\partial g_{53}}{\partial x^1} \right) + \frac{1}{2}g^{16} \frac{\partial g_{56}}{\partial x^3} \\ &= \frac{1}{2} \left( \frac{\partial A_1}{\partial x^2} - \frac{\partial A_3}{\partial x^1} \right) + \frac{1}{2}Z_1 \frac{\partial g_{56}}{\partial x^3}, \end{aligned} \quad (\text{A15})$$

$$\Gamma_{54}^1 = \frac{1}{2}g^{11} \left( \frac{\partial g_{51}}{\partial x^4} - \frac{\partial g_{54}}{\partial x^1} \right) + \frac{1}{2}g^{16} \frac{\partial g_{56}}{\partial x^4} = \frac{1}{2} \left( \frac{\partial A_1}{\partial x^4} + \frac{1}{c} \frac{\partial \phi}{\partial x^1} \right); \quad (\text{A16})$$

and

$$\Gamma_{61}^1 = \frac{1}{2}g^{11} \left( \frac{\partial g_{61}}{\partial x^1} - \frac{\partial g_{61}}{\partial x^1} \right) + \frac{1}{2}g^{16} \frac{\partial g_{66}}{\partial x^1} = 0, \quad (\text{A17})$$

$$\begin{aligned} \Gamma_{62}^1 &= \frac{1}{2}g^{11} \left( \frac{\partial g_{61}}{\partial x^2} - \frac{\partial g_{62}}{\partial x^1} \right) + \frac{1}{2}g^{15} \frac{\partial g_{65}}{\partial x^2} \\ &= \frac{1}{2} \left( \frac{\partial Z_1}{\partial x^2} - \frac{\partial Z_2}{\partial x^1} \right) + \frac{1}{2}A_1 \frac{\partial g_{65}}{\partial x^2}, \end{aligned} \quad (\text{A18})$$

$$\begin{aligned} \Gamma_{63}^1 &= \frac{1}{2}g^{11} \left( \frac{\partial g_{61}}{\partial x^3} - \frac{\partial g_{63}}{\partial x^1} \right) + \frac{1}{2}g^{15} \frac{\partial g_{65}}{\partial x^3} \\ &= \frac{1}{2} \left( \frac{\partial Z_1}{\partial x^2} - \frac{\partial Z_3}{\partial x^1} \right) + \frac{1}{2}A_1 \frac{\partial g_{65}}{\partial x^3}, \end{aligned} \quad (\text{A19})$$

$$\Gamma_{64}^1 = \frac{1}{2}g^{11} \left( \frac{\partial g_{61}}{\partial x^4} - \frac{\partial g_{64}}{\partial x^1} \right) + \frac{1}{2}g^{15} \frac{\partial g_{56}}{\partial x^4} = \frac{1}{2} \left( \frac{\partial Z_1}{\partial x^4} + \frac{1}{c} \frac{\partial \eta}{\partial x^1} \right). \quad (\text{A20})$$

The time independence of  $g_{56}$  is justified by equations (A16) and (A20). For example, the symbols calculated in Eq. (A16) should be  $-E_1/c$ , which, considering the traditional definition for the electric field in terms of potentials should read  $(\partial A_1/\partial x^4 + \partial(\phi/c)/\partial x^1)$ .

Comparing equations (A5) to (A12) and (A13) to (A20) the relations between the fields and the potentials can be expressed as follows

$$E^\nu = -\frac{\partial \phi}{\partial x^\nu} - \frac{\partial A_\nu}{\partial t}, \quad (\text{A21})$$

$$E^\nu = \varepsilon_{\beta\gamma}^\nu \left( \frac{\partial Z_\beta}{\partial x^\gamma} - \frac{\partial Z_\gamma}{\partial x^\beta} \right) + M^\nu, \quad (\text{A22})$$

$$B^\nu = -\frac{\partial \eta}{\partial x^\nu} - \frac{\partial Z_\nu}{\partial t}, \quad (\text{A23})$$

$$B^\nu = \varepsilon_{\beta\gamma}^\nu \left( \frac{\partial A_\beta}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^\beta} \right) + Q^\nu. \quad (\text{A24})$$

The reason for having two equivalent expressions for each field in terms of potentials comes from a duality in Faraday's tensor which can be expressed in terms of the equivalence in Christoffel symbols with indices 5 and 6. For example,

$$\Gamma_{64}^1 = -\frac{1}{c}B_1 = \Gamma_{52}^3. \quad (\text{A25})$$

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